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ON THE RANGES OF ANALYTIC FUNCTIONS

BY

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ABSTRACT. Following Doob, we say that a function $f(z)$ analytic in the unit disk U has the property $K(\rho)$ if $f(0) = 0$ and for some arc A on the unit circle whose measure $|A| \geq 2\rho > 0$,

$$\liminf_{i\to\infty} |f(P_i)| \geq 1 \quad \text{where } P_i \to P \in A \text{ and } P_i \in U.$$

We recently have solved a problem of Doob by showing that there is an integer $N(\rho)$ such that no function with the property $K(\rho)$ can satisfy

$$(1 - |z|)|f_n'(z)| \leq 1/n \quad \text{for } z \in U, \text{ where } n > N(\rho).$$

The function

$$f_n(z) = 1 + (1 - z^n)/n^2,$$

shows that the condition $f_n(0) = 0$ is necessary and cannot be replaced by $f_n(0) = re^{i\alpha}$, for $r > 1$. Naturally, we may ask whether this can be replaced by $f_n(0) = re^{i\alpha}$, for $r < 1$? The answer turns out to be yes, when $n > N(r, \rho)$, where

$$N(r, \rho) \doteqdot (1/(1 - r))\log(1/(1 - \cos\rho)).$$

1. Introduction. Let $f(z)$ be a function analytic in the unit disk $U = \{z: |z| < 1\}$. Following Doob [3, p. 119], we say that a function $f(z)$ has the property $K(\rho)$ if $f(0) = 0$ and for some arc A on the unit circle $C = \{z: |z| = 1\}$ whose measure $|A| \geq 2\rho > 0$,

$$\liminf_{i\to\infty} |f(P_i)| \geq 1, \tag{1}$$

where $\{P_i\}$ is a sequence of points in U tending to a point on A.

In a recent paper [6], we have solved a problem of Doob by showing that there is an integer $N(\rho)$ such that no function $f_n(z)$ with the property $K(\rho)$ can satisfy

$$(1 - |z|)|f_n'(z)| \leq 1/n \quad \text{for } z \in U, \text{ where } n > N(\rho). \tag{2}$$

In this result, the normalization $f_n(0) = 0$ is necessary and cannot be replaced by $f_n(0) = re^{i\alpha}$, for $r > 1$, as will be seen from the following

EXAMPLE 1. The function

$$f_n(z) = 1 + (1 - z^n)/n^2,$$

satisfies both conditions (1) and (2), but $f_n(0) > 1$.

Naturally, we may ask whether this condition $f_n(0) = re^{i\alpha}$, $r > 1$, can be replaced by $r < 1$? The answer turns out to be yes.

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THEOREM 1. *Let $K_n(r, \rho)$ be the class of all functions $f_n(z)$ satisfying* (1), (2) *and* $|f_n(0)| \leqslant r < 1$. *Then this class is empty provided $n > N(r, \rho)$, where*

$$N(r, \rho) \doteqdot (1/(1-r))\log(1/(1-\cos\rho)) \quad \text{as } \rho \to 0. \tag{3}$$

2. Proof of Theorem 1. The method here is the same as that of [6, Theorem 1] and therefore we sketch the details. First of all, we assume that the arc A is symmetric with respect to the point at $z = 1$ and we let G be the domain bounded by A and its subtended chord. Then clearly G is a moon shape domain such that the angles formed at the two endpoints of its boundary are the same, which is equal to ρ.

Now suppose, on the contrary, that this class $K_n(r, \rho)$ is not empty for some $n > N(r, \rho)$ and let

$$f_n \in K_n(r, \rho) \quad \text{and} \quad g(z) = 1/f_n(z) \quad \text{for } z \in U.$$

Let $\varepsilon > 0$ be given; then by the same argument as [6], we obtain

$$|g(z)| \leqslant \lambda \quad \text{or} \quad |f_n(z)| \geqslant 1/\lambda \quad \text{for } z \in G, \tag{4}$$

where λ is the smallest positive solution of

$$h(\lambda) = (1+\varepsilon)\exp[(\rho/n \sin\rho)(\lambda + 1/\lambda)] - \lambda. \tag{5}$$

It follows from (5) that $h(1) > 0$ and there is an integer $N(\varepsilon)$ such that

$$h(1+2\varepsilon) < 0 \quad \text{for } n > N(\varepsilon).$$

This gives the estimate

$$1 < \lambda < 1 + 2\varepsilon. \tag{6}$$

Let P be the intersection of the chord of G with the segment $[0, 1)$; then by (4) and (6), we find that

$$|f_n(P)| > 1/(1+2\varepsilon). \tag{7}$$

On the other hand, from condition (2) and the hypothesis, we get

$$|f_n(z)| \leqslant \left| \int_0^{|z|} f_n'(w)\, dw \right| + |f_n(0)| \leqslant \int_0^{|z|} \frac{1}{n(1-x)}\, dx + r$$
$$= \frac{1}{n} \log \frac{1}{1-|z|} + r. \tag{8}$$

In view of (3), there is a number δ with $0 < \delta < 1 - r$ such that

$$(1/n)\log(1/(1-\cos\rho)) < 1 - r - \delta \quad \text{for } n > N(r, \rho). \tag{9}$$

Since the point P is located at $\cos\rho$, hence by (8) and (9), we obtain

$$|f_n(P)| < 1 - r - \delta + r = 1 - \delta \quad \text{for } n > \max(N(\varepsilon), N(r, \rho)).$$

This, however, contradicts (7) provided $\varepsilon < \delta/2(1-\delta)$.

Clearly, we have $\max(N(\varepsilon), N(r, \rho)) = N(r, \rho)$ as $\rho \to 0$ and therefore we conclude the result.

Notice that the above theorem holds for any $\rho > 0$ provided $N(r, \rho)$ is replaced by $eN(r, \rho)$, see [6, Theorem 1].

3. The ranges of $K_n(r, \rho)$-class. By the same argument as [**6**, Theorem 2], it is not difficult to prove the following covering property for functions in the $K_n(r, \rho)$-class, where $n < N(r, \rho)$.

THEOREM 2. *If $f(z)$ is a function in the $K_n(r, \rho)$-class, then the range of $f(z)$ covers the interior of some circle of radius*

$$k(r, \rho) = 1/16N(r, \rho) \quad \text{where } N(r, \rho) \text{ is defined in } (3).$$

Notice that between the positive theorems and negative example for the conditions of $r < 1$ and $r > 1$ respectively, we may ask what can be said about $r = 1$? In this case, we have $N(r, \rho) = \infty$ and $k(r, \rho) = 0$, and, in fact, both theorems are no longer true due to the following function.

$$f_n^*(z) = 1 + z^n/n^2 \quad \text{where } f_n^*(0) = r = 1.$$

4. The ranges of $K(\rho, s)$-class. Instead of the $K(\rho)$ property of Doob, we now consider the more general $K(\rho, s)$ property, namely, keep everything the same except for (1) being replaced by

$$\liminf_{i\to\infty} |f(P_i)| \geqslant s > 0 \quad \text{where } P_i \in U. \tag{10}$$

Thus the $K(\rho)$ property = the $K(\rho, 1)$ property. Clearly, if a function f has the $K(\rho, s)$ property, then the function $g(z) = f(z)/s$ has the $K(\rho)$ property. It is geometrically clear that if the range of $g(z)$ covers the interior of some circle of radius $k(\rho)$, then the corresponding covering radius of $f(z)$ will be $sk(\rho)$. This observation together with [**6**, Theorem 2] yields

THEOREM 3. *If $f(z)$ has the $K(\rho, s)$ property, then the range of $f(z)$ covers the interior of some circle of radius*

$$k(\rho, s) = s/16e \log(1/(1 - \cos\rho)).$$

5. The ranges of $K(r, \rho, s)$-class. We have already defined the $K(r, \rho)$ and $K(\rho, s)$ property. We now define the $K(r, \rho, s)$ property to be the common property of $K(r, \rho)$ and $K(\rho, s)$. Combining Theorems 2 and 3, we can easily obtain the following general result.

THEOREM 4. *If $f(z)$ has the $K(r, \rho, s)$ property, then the range of $f(z)$ covers the interior of some circle of radius*

$$k(r, \rho, s) = s/16e(1/(1 - r))\log(1/(1 - \cos\rho)),$$

where $r < 1$ and $0 < \rho < \pi/6$.

For the purpose of next two sections, we shall now give the boundary behaviour of functions in $K(r, \rho, s)$-class.

THEOREM 5. *If $f(z)$ has the $K(r, \rho, s)$ property, then $f(z)$ has radial as well as angular limits almost everywhere on the arc A.*

PROOF. Let G be the domain bounded by A and its subtended chord. Then by condition (10), we can see that the cluster set of $f(z)$ over the domain G is not total.

It follows that for some value v, the function

$$g(z) = 1/(f(z) - v) \quad \text{for } z \in U,$$

is bounded in G.

Denote $z(w)$ to be the conformal mapping from U onto G. Then the function

$$h(w) = g(z(w)) \quad \text{for } w \in U,$$

is bounded in U. Therefore by Fatou and Lindelöf's theorem, see [**2**, Theorem 2.4], the function $h(w)$ has radial as well as angular limits almost everywhere on C. The conclusion now follows from the Riesz theorem [**2**, Theorem 3.3].

6. The ranges of $K(0)$-class. All results in the previous sections are in a sense sufficient but not necessary. In fact, as far as the range is concerned, the opposite version is stronger than the positive one.

For instance, in each theorem we require the function $f(z)$ to be holomorphic in U in order to guarantee that the range of $f(z)$ covers the interior of some circle of radius $r > 0$. But in the opposite version, if $f(z)$ has a pole, then the range covers the whole extended plane outside a finite circle and of course, the corresponding radius $r = \infty$.

From this observation, we can see that the complement of our conditions, in a sense, is not only not worse, but also better than our propositions. We shall now explain the same phenomenon in terms of a different class of functions. For this we state that a function $f(z)$ has the $K(0)$ property if $f(0) = 0$, and $f(z)$ satisfies condition (2), and furthermore, the function $f(z)$ has no radial limits everywhere on C.

In view of Theorem 5, we can see that the $K(0)$-class is inside the complement of the $K(r, \rho, s)$-class. In this case, not only the result of Theorem 4 is true, but also much better than that of Theorem 4. Before proving the next theorem, we shall show that this $K(0)$-class is not empty. Notice that if $f \in K(0)$, then (10) can hold only for $s = 0$.

EXAMPLE 2. The function

$$f(z) = \frac{1}{2n} \sum_{i=0}^{\infty} z^{2^i} \quad \text{for } z \in U,$$

satisfies $f(0) = 0$ and condition (2) because, for $|z| = r$,

$$|f'(z)| \leq \frac{1}{n}(1 + r + 2r^3 + 4r^7 + \cdots) \leq \frac{1}{n}\frac{1}{1-r} = \frac{1}{n}\frac{1}{1-|z|}.$$

Moreover, by virtue of Hardy and Littlewood's theorem [**4**], we can see that this gap series $f(z)$ cannot have Abel sum, i.e. has no radial limits everywhere on C. Hence the function $f \in K(0)$.

We shall now turn to prove our theorem. For convenience, we denote a sector by $\delta(\theta_1, \theta_2)$ if it is bounded by two radii with endpoints at $e^{i\theta_j}, j = 1, 2$.

THEOREM 6. *If $f(z)$ has the $K(0)$ property, then the range of $f(z)$ over any sector $\delta(\theta_1, \theta_2)$ covers the whole plane.*

PROOF. Suppose on the contrary that there is an omitted value v. We shall show that the function $f(z)$ will have the radial limit v at a point $e^{i\theta}$ where $\theta_1 < \theta < \theta_2$. The method here is the same as that of [5] and therefore we sketch it.

From condition (2), we know that $f(z)$ is a normal function, see Lehto and Virtanen [8]. It follows from a theorem of Bagemihl and Seidel [1, Theorem 3] that there is a dense subset S of C such that the function $f(z)$ tends to infinity at any point of S. Choose any two points of this kind, say $e^{i\alpha_j}$, $\theta_1 < \alpha_1 < \alpha_2 < \theta_2$. Then there can be constructed a path β lying on $\delta(\alpha_1, \alpha_2)$ such that $f(z)$ tends to v along this path β.

Owing to [1, Theorem 1], we find that this path β must end at a point $e^{i\alpha}$, $\alpha_1 < \alpha < \alpha_2$. It follows from [8, Theorem 2] that the function $f(z)$ has the angular limit v at $e^{i\alpha}$, which is a contradiction.

7. The ranges of Doob's class. As was introduced in [7], a function $f(z)$ belongs to Doob's class D if it satisfies (1) on an arc A and $f(z)$ has radial limit 0 at an endpoint of A. Instead of the limit 0, we shall now call a function $f \in D(a)$, where $|a| < 1$, if in addition to (1), $f(z)$ has radial limit a at an endpoint of A. With this notion, we are going to prove the following general result.

THEOREM 7. *If $f \in D(a)$, where $|a| < 1$, then the range of $f(z)$ covers the interior of some circle of radius $k(a) = (1 - |a|)/8e$.*

PROOF. Clearly, the following function $g \in D$,

$$g(z) = (f(z) - a)/(1 - |a|) \quad \text{for } z \in U.$$

It follows from [7, Theorem 1] that the following Bloch norm of g satisfies

$$\|g\| = \sup_{z \in U} |g'(z)|(1 - |z|^2) \geqslant 2/e,$$

which in turn implies $\|f\| \geqslant 2(1 - |a|)/e$. By the same argument as [6, Theorem 2], we can easily obtain the result.

Notice that if the case $|a| = 1$ is allowed in Doob's class, then the radius $k(a) = 0$. In this case, the above Theorem 7 cannot be improved to be $k(a) > 0$; see Example 1.

8. The ranges of $D^*(a)$-class. In this section, we shall consider another kind of function. For this, we say that a function $f \in D^*(a)$, where $|a| < 1$, if condition (1) is satisfied through an arc A except at an interior point P of A such that $f(z)$ has the radial limit a at this point P. Functions of this kind have range independent of the number a as will be seen from the following.

THEOREM 8. *If $f \in D^*(a)$, where $|a| < 1$, then the range of $f(z)$ covers the interior of some circle of radius $1/2$.*

PROOF. We may assume that the interior point is located at $P = 1$ for which $f(z)$ tends to a as $z \to 1^-$. Let λ be a number with $|a| < \lambda < 1$ and let G_λ be the subdomain of U defined by

$$G_\lambda = \{z\colon |f(z)| \geqslant \lambda\} \quad \text{where } |a| < \lambda < 1.$$

We shall first prove that the point P lies on the closure $\bar{G}_\lambda$. To see this, let $L(Q)$ be a chord starting from P and ending at a point $Q \in A$. Then by Theorem 5, we know that $f(z)$ tends to a when z tends to P along $L(Q)$. In view of (1), we have $|f(Q)| \geqslant 1$. By the continuity of modulus, we find that

$$|f(P^\lambda)| = \lambda \quad \text{for some point } P^\lambda \in L(Q).$$

Clearly, by choosing a sequence $Q_n \to P$, where $Q_n \in A$, we obtain a sequence

$$P_n^\lambda \to P \quad \text{where } P_n^\lambda \in \bar{G}_\lambda \text{ for } n = 1, 2, \ldots. \tag{11}$$

This shows that $P \in \bar{G}_\lambda$.

Let ∂G_λ be the boundary of G_λ, which is the level set of $f(z)$ at the level λ. We now choose $\lambda > 1/2$ and we shall prove our assertion by considering two cases: either the level set ∂G_λ contains a component in U or not. For the first case, we let ∂H be a component of ∂G_λ contained in U. Then ∂H contains a closed Jordan curve which bounds a domain, say, H and satisfies $|f(z)| < \lambda$ for $z \in H$. By the minimum principle, we can see that H contains some zeros of $f(z)$. Since $\lambda > 1/2$, the range of $f(z)$ over H covers the disk with center at the origin and radius $1/2$.

On the other hand, each component of ∂G_λ must end somewhere on C. By what we have shown in (11), there is a Jordan arc J lying in U for which $P \in J$. Since $\lambda < 1$, from condition (1), we find that this arc J must end at P. Notice that the point $P = 1$ is an interior point of A which separates A into an upper and lower part. Therefore, in the second case, we can find two Jordan arcs J_1 and J_2 lying on the upper and lower part with respect to the radius $[0, 1)$ and ending at this common point P. Let P_i be the other endpoints of J_i different from P, $i = 1, 2$. Clearly, P_1 and P_2 can be joined by a suitable arc J_3 lying in U such that the domain G bounded by J_i, $i = 1, 2, 3$, satisfies

$$\begin{aligned} |f(z)| &\leqslant \lambda \quad \text{for all } z \in G \subset U, \\ &= \lambda \quad \text{for } z \in J_1 \cup J_2. \end{aligned} \tag{12}$$

We now consider the conformal mapping $z(w)$ from U onto G satisfying $z(1) = 1$. Then the inverse $A^* = w(J_1 \cup J_2)$ is an arc on C containing the point $P = 1$ in its interior. We set the function

$$g(w) = f(z(w))/\lambda \quad \text{for } w \in U.$$

It follows from (12) that

$$\begin{aligned} |g(w)| &\leqslant 1 \quad \text{for } w \in U, \\ &= 1 \quad \text{for } w \in A^*. \end{aligned}$$

Moreover, the function $g(w)$ has the radial limit a as $w \to 1^-$. Hence the point P is an isolated singularity of $g(w)$. By virtue of [7, Theorem 7], we find that since the range of $g(w)$ over U covers the interior of some circle of radius $1/2$, so does the function $f(z)/\lambda$ over G. Thus the range of $f(z)$ over G covers the interior of some circle of radius $\lambda/2$. By letting $\lambda \to 1$, we obtain the desired result.

9. Remark. To end this paper, let us pose two conjectures concerning the $K(0)$-class.

CONJECTURE 1. If $f \in K(0)$, then the cluster set $C(f, e^{i\theta})$ is total, i.e. the extended plane, for any θ.

Notice that if $f \in K(0)$, then by [**1**], we know that the set of Fatou points of f with value ∞ is dense and of measure zero. It follows from Plessner's theorem [**2**, Theorem 8.2] that the cluster set $C(f, e^{i\theta})$ is total for almost all θ. Therefore, Conjecture 1 needs to be proved only for those Fatou points. In other words, if $e^{i\theta}$ is a Fatou point, then the tangential cluster set should be total. Of course, the nontangential cluster set is the value ∞.

CONJECTURE 2. If $f(z) = \sum a_n z^n \in K(0)$, then

$$\liminf_{n\to\infty} |a_n| = A_i = 0 \quad \text{and} \quad \limsup_{n\to\infty} |a_n| = A_s > 0.$$

In view of Example 2, the function $f(z)$ defined there satisfies $A_i = 0$ and $A_s = 1 > 0$. This function should represent the general property of coefficients of the $K(0)$-class. Of course, if the function $f(z)$ has no radial limits almost everywhere instead of everywhere, then the number A_s can be zero.

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